

Coefficient of performance under maximum χ criterion in a two-level atomic system as a refrigerator

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Abstract

A two-level atomic system as a working substance is used to set up a refrigerator consisting of two quantum adiabatic and two isochoric processes (two constant-frequency processes ω_a and ω_b with $\omega_a < \omega_b$), during which the two-level system is in contact with two heat reservoirs at temperatures T_h and $T_c (< T_h)$. Considering finite-time operation of two isochoric processes, we derive analytical expressions for cooling rate R and coefficient of performance (COP) ε . The COP at maximum $\chi (= \varepsilon R)$ figure of merit is numerically determined, and it is proved to be in nice agreement with the so-called Curzon and Ahlborn COP $\varepsilon_{CA} = \sqrt{1 + \varepsilon_C} - 1$, where $\varepsilon_C = T_c/(T_h - T_c)$ is the Carnot COP. In the high-temperature limit, the COP at maximum χ figure of merit, ε^* , can be expressed analytically by $\varepsilon^* = \varepsilon_+ \equiv (\sqrt{9 + 8\varepsilon_C} - 3)/2$, which was derived previously as the upper bound of optimal COP for the low-dissipation or minimally nonlinear irreversible refrigerators. Within context of irreversible thermodynamics, we prove that the value of ε_+ is also the upper bound of COP at maximum χ figure of merit when we regard our model as a linear irreversible refrigerator.

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I. INTRODUCTION

A heat device is a heat engine that convert thermal energy into mechanical work, or a refrigerator (heat pump) that is basically a heat engine running backwards. For an endoreversible heat engine working between a hot and a cold reservoir at constant temperatures T_h and $T_c (< T_h)$, Curzon and Ahlborn (CA) [1] found the efficiency of at maximum power to be $\eta_{CA} = 1 - \sqrt{T_c/T_h} = 1 - \sqrt{1 - \eta_C}$ with $\eta_C = 1 - T_c/T_h$ the Carnot efficiency. The model of such a heat engine presented by Curzon and Ahlborn gave rise to the birth and intensive studies of finite-time thermodynamics [2–14], a branch of thermodynamics focusing on the optimization on the energy converter that consists of some finite-time thermodynamic processes. The universality and bounds [12, 15–23] of the efficiency at maximum power have been discussed in a large number of studies of heat engines within the context of finite-time thermodynamics.

Unlike in analysis of a heat engine where the power output is always an objective function to determine the optimized efficiency, there are various optimization criteria [24–31] in analysis of optimization of a refrigerator working between two heat reservoirs with constant temperatures T_h and T_c . One of these criteria for a refrigerator, which was first proposed by Yan and Chen [24], is taking the target function $\chi = \varepsilon Q_c/t_{cycle}$, where Q_c is heat absorbed from the cold reservoir, t_{cycle} denotes the cycle time, and $\varepsilon = Q_c/W$ with W being the work input per cycle is the coefficient of performance (COP) for refrigerators. This χ -optimization criterion for refrigerators is always adopted and found to be exactly the counterpart [27, 28, 31] for the optimization of power output for heat engines. For a low dissipation [30, 31] or a minimally nonlinear irreversible [29] refrigerator, the lower and upper bounds of the COP at maximum χ figure of merit (ε^*) have been found to be: $0 \leq \varepsilon^* \leq (\sqrt{9 + 8\varepsilon_C} - 3)/2$, with $\varepsilon_C = T_c/(T_h - T_c)$ being so-called Carnot COP.

The research into heat engines or refrigerators has been extend from classical to quantum systems [11, 14, 32–49] over 50 years. This is motivated by exploring the emergence of basic thermodynamic description at the quantum mechanical level, and also by the potential technological applications of these devices [39, 42, 44, 45, 50]. In particular, demands for smaller heat devices have been rapidly rising because of miniaturization in experiment [39, 50] and understanding of quantum thermodynamics [48, 49]. The ongoing reduction in system size is approaching the ultimate limit, scaling downing these heat devices to a single

particle system, in which quantum properties become significant and have thus to be fully considered.

As quantum versions of classical thermodynamic cycles, quantum thermodynamic cycles have also a set of different cyclic heat device models [37] working between heat reservoirs. The quantum Otto cycle, which is a typical one of these models and the quantum analog of the classical Otto cycle used widely in practical heat devices, has been proposed and discussed in a series of papers [11, 14, 36–38, 42]. The present paper employs a two-level atomic system as a working substance to set up a refrigerator model, which consists of two isochoric and two adiabatic processes and is thus a quantum version of the Otto refrigeration cycle. Based on master equations of stochastic processes, we derive expressions for the cooling rate and power input, which are functions of the time allocation on the two isochores. The objective function χ is then numerically optimized to determine the optimal COP ε^* , which is also analytically expressed as a function of Carnot COP ε_C in the high-temperature limit. Finally, we analyze the COP at maximum χ figure of merit by taking our model as a linear irreversible refrigerator satisfying the tight-coupling condition.

II. A MODEL OF QUANTUM OTTO REFRIGERATION CYCLE

A. Dynamics of occupation probabilities

In the refrigerator model the working substance is a two-level energy system, with ground state g and excited state e characterized by the energy spectrum $\varepsilon_g = \omega$ and $\varepsilon_e = 2\omega$ ($\hbar \equiv 1$) and by the energy gap $\Delta\varepsilon = \varepsilon_e - \varepsilon_g = \omega$. Let p_g and p_e denote the occupation probabilities of the two states e and g , and these probabilities must satisfy the constraint $p_e + p_g = 1$.

When a two-level energy system is coupled to a heat reservoir at constant temperature $T = 1/\beta$ ($k_B \equiv 1$), the dynamics of the occupation probabilities at the ground and excited states, p_g and p_e , can be determined according to the following master equation [38, 47]:

$$\dot{\mathbf{p}}(t) = R \cdot \mathbf{p}(t), \quad (1)$$

with $\mathbf{p}(t) = (p_g, p_e)^T$ (where the superscript T denotes transpose). Here the stochastic matrix R describing particle dynamics is given by

$$R = \begin{pmatrix} -k_{\uparrow} & k_{\downarrow} \\ k_{\uparrow} & -k_{\downarrow} \end{pmatrix}, \quad (2)$$

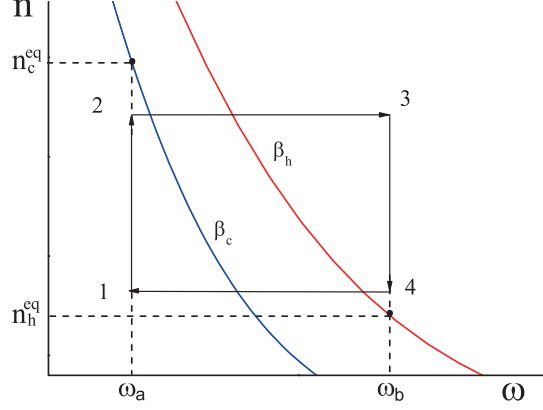


FIG. 1: (Color online) Schematic diagram of a quantum Otto refrigeration cycle in the (ω, n) plane. $1 \rightarrow 2$ and $3 \rightarrow 4$ are two isochoric processes, while $2 \rightarrow 3$ and $4 \rightarrow 1$ are two adiabatic processes. n_h^{eq} and n_c^{eq} are two ratios when the atomic system achieves thermal equilibrium with two heat reservoirs at inverse temperatures β_h and β_c , respectively.

where k_\downarrow and k_\uparrow represent the transition rates from the excited to the ground level and vice versa, and they satisfy the requirement of detailed balance [49, 51]:

$$\frac{k_\uparrow}{k_\downarrow} = e^{-\beta\Delta\varepsilon}. \quad (3)$$

Then these transition rates can be parameterized by [49]

$$k_\downarrow = \gamma(1 - \sigma), k_\uparrow = \gamma(1 + \sigma), \quad (4)$$

with $\sigma = \tanh(\beta\Delta\varepsilon/2)$, where $\gamma > 0$ denotes a characteristic rate for these transitions and it will be identified as the heat conductivity in the following [below Eq. (13)].

Now we turn to the discussion of the refrigerator model operating in finite time. The working substance of the quantum Otto refrigerator is a two-level atomic system with time-dependent energy unit $\omega(t)$, changing between ω_a and ω_b . The two level system is alternately coupled to two heat baths at inverse temperatures β_c and $\beta_h (< \beta_c)$. The Otto cycle consists of four consecutive steps shown in Fig. 1 and it is described as follows:

(1) Cold isochore $1 \rightarrow 2$. Initially at time $t = 0$, the system becomes coupled to a cold reservoir at inverse temperature β_c and is decoupled from this reservoir until time $t = \tau_c$, while the frequency is kept constant ω_a . From Eq. (1), we obtain the probabilities $\mathbf{p}(t)$ at any instant of the isochore ($0 \leq t \leq \tau_c$) as,

$$\mathbf{p}(t) = \exp(R_c t) \mathbf{p}(0) \quad (5)$$

where $R_c = \gamma_c \begin{pmatrix} -(1 + \sigma_c) & (1 - \sigma_c) \\ (1 + \sigma_c) & -(1 - \sigma_c) \end{pmatrix}$ with $\sigma_c = \tanh(\beta_c \omega_a/2)$.

(2) Adiabatic compression $2 \rightarrow 3$. The system is isolated in time τ_{ch} while its frequency ω changes from ω_a to ω_b , with a slow speed to satisfy the quantum adiabatic condition. In the adiabatic process, the entropy is kept constant as

$$\mathbf{p}(t) = \mathbf{p}(\tau_c), \quad (6)$$

with $\tau_c \leq t \leq \tau_c + \tau_{ch}$.

(3) Hot isochore $3 \rightarrow 4$. The system is now coupled to a hot reservoir at inverse temperature $\beta_h (< \beta_c)$ in time of τ_h and its energy unit is again kept constant. As in the process $1 \rightarrow 2$, the evolution of the probabilities $\mathbf{p}(t)$ at any instant ($\tau_c + \tau_{ch} \leq t \leq \tau_c + \tau_{ch} + \tau_h$) during the hot isochore can be determined according to

$$\mathbf{p}(t) = \exp(R_h t) \mathbf{p}(\tau_c + \tau_{ch}), \quad (7)$$

where $R_h = \gamma_h \begin{pmatrix} -(1 + \sigma_h) & (1 - \sigma_h) \\ (1 + \sigma_h) & -(1 - \sigma_h) \end{pmatrix}$ with $\sigma_h = \tanh(\beta_h \omega_b/2)$.

(4) Adiabatic expansion $4 \rightarrow 1$. The energy unit ω is changed very slowly (as in the adiabatic compression) to its initial value ω_a , while the probabilities \mathbf{p} are kept unchanged. Let τ_{hc} be the time taken for completing this adiabat. When $\tau_c + \tau_{ch} + \tau_h \leq t \leq \tau_c + \tau_{ch} + \tau_h + \tau_{hc}$, we have

$$\mathbf{p}(t) = \mathbf{p}(\tau_c + \tau_{ch} + \tau_h). \quad (8)$$

After a single cycle, the entropy of the system as a state function changes back to its initial value, and therefore we have $\mathbf{p}(t_{cycle}) = \mathbf{p}(0)$, where $t_{cycle} \equiv \tau_c + \tau_{ch} + \tau_h + \tau_{hc}$ is the cycle time. It follows, using Eqs. (5)- (8), that the probabilities of the final and initial system states during a cycle satisfy the relation:

$$\begin{pmatrix} p_g(t_{cycle}) \\ p_e(t_{cycle}) \end{pmatrix} = \exp(R_h t) \exp(R_c t) \begin{pmatrix} p_g(0) \\ p_e(0) \end{pmatrix}. \quad (9)$$

where R_c and R_h were defined in Eqs. (5) and (7), respectively. Let $\mathcal{M} = \exp(R_h t) \exp(R_c t)$ be the transition matrix for the two-level system proceeding a cycle. Note that, the initial instant of the system per cycle under consideration can be assumed to be a periodic steady

state [52]. Considering Eq. (9), we find

$$\begin{pmatrix} p_g(0) \\ p_e(0) \end{pmatrix} = \mathcal{M} \begin{pmatrix} p_g(0) \\ p_e(0) \end{pmatrix}, \quad (10)$$

from which we obtain the probabilities $\mathbf{p}(0)$ at the initial instant per cycle as

$$\mathbf{p}(0) = \begin{pmatrix} p_g(0) \\ p_e(0) \end{pmatrix} = \begin{bmatrix} \frac{(1-\sigma_h)e^{2(\gamma_c\tau_c+\gamma_h\tau_h)}+(\sigma_h-\sigma_c)e^{2\gamma_c\tau_c+\sigma_c-1}}{2e^{2(\gamma_c\tau_c+\gamma_h\tau_h)}-2} \\ \frac{(\sigma_h+1)e^{2(\gamma_c\tau_c+\gamma_h\tau_h)}+(\sigma_c-\sigma_h)e^{2\gamma_c\tau_c-\sigma_c-1}}{2e^{2(\gamma_c\tau_c+\gamma_h\tau_h)}-2} \end{bmatrix}. \quad (11)$$

Using Eqs. (5) and (11), we obtain

$$\mathbf{p}(\tau_c) = \begin{pmatrix} p_g(\tau_c) \\ p_e(\tau_c) \end{pmatrix} = \begin{bmatrix} \frac{(1-\sigma_c)e^{2(\gamma_c\tau_c+\gamma_h\tau_h)}+(\sigma_c-\sigma_h)e^{2\gamma_h\tau_h+\sigma_h-1}}{2e^{2(\gamma_c\tau_c+\gamma_h\tau_h)}-2} \\ \frac{(\sigma_c+1)e^{2(\gamma_c\tau_c+\gamma_h\tau_h)}+(\sigma_h-\sigma_c)e^{2\gamma_h\tau_h-\sigma_h-1}}{2e^{2(\gamma_c\tau_c+\gamma_h\tau_h)}-2} \end{bmatrix}. \quad (12)$$

B. Cooling load and COP

The amount of change in energy dE for the system follows from the first law of thermodynamics: $dE = dQ + dW = \sum_\nu \varepsilon_\nu dp_\nu + \sum_\nu p_\nu d\varepsilon_\nu$, where $dQ = \sum_\nu \varepsilon_\nu dp_\nu$ and $dW = \sum_\nu p_\nu d\varepsilon_\nu$ are the heat exchange and work done, respectively. Accordingly, during an adiabatic process there is no heat exchange ($dQ = 0$) as the occupation probabilities p_ν do not change, but work may still be nonzero (since eigenenergies ε_ν may change). For simplicity, the total energy of the two-level system ($E = p_g\omega + 2p_e\omega$) can be written as $E = n\omega$, with $n \equiv p_g + 2p_e$. It follows, using the relation $n = p_g + 2p_e$, that

$$dE = dW + dQ = nd\omega + \omega dn, \quad (13)$$

where $dQ = \omega dn$ and $dW = nd\omega$. The energy of the system can change either by particle transition (changing n) or by varying the energy gap between two states (changing ω). The heat current during the hot or cold isochoric process is determined by $dQ = \omega \frac{dn}{dt} = \frac{d(p_g+2p_e)}{dt}$ which, together with Eqs. (1), (2) and (4), indicates that $\gamma_{c,h}$ represent the heat conductivities between the working substance and the cold and hot reservoirs, respectively.

Since there exist no heat exchanged during the two adiabatic process, we can merely determine the heat exchanged during the cold and hot isochoric processes, Q_c and Q_h , to obtain the COP, $\varepsilon = Q_c/W$ with work input W . In view of the fact that no work is done in any isochore, the heat absorbed by the system from the cold reservoir in the cold isochore

$1 \rightarrow 2$, which is just the cooling load, can be directly calculated as

$$Q_c = \int_0^{\tau_c} \omega_a \frac{dn}{dt} dt = (n_2 - n_1) \omega_a, \quad (14)$$

where $n_2 = p_g(\tau_c) + 2p_e(\tau_c)$ and $n_1 = p_g(0) + 2p_e(0)$. Similarly, we can easily derive the amount of heat released to the hot reservoir during the hot isochore $3 \rightarrow 4$ as

$$Q_h = \left| \int_{\tau_c + \tau_{ch}}^{\tau_c + \tau_{ch} + \tau_h} \omega_b \frac{dn}{dt} dt \right| = (n_2 - n_1) \omega_b, \quad (15)$$

where the use of $n_3 = n_2$ and $n_4 = n_1$ has been made (see Fig. 1). Then the COP of the quantum refrigerator becomes

$$\varepsilon = \frac{\omega_a}{\omega_b - \omega_a}. \quad (16)$$

As shown in Eqs. (6) and (8), the occupation probabilities $\mathbf{p}(t)$ are kept unchanged during an adiabatic process. It follows, using Eqs. (11) and (12) as well as $T = 1/\beta$, that in the adiabatic process there exists the relation $T\omega^{-1} = \text{const}$. Making a comparison of this adiabat relation to that for classical ideal gas, $TV^{\gamma-1} = \text{const}$, with γ the adiabatic parameter, we find that the Otto cycle COP (16) for the two level system is analogous to the COP of the Otto cycle working with the classical ideal gas: $\varepsilon^{IG} = \left(\frac{V_a}{V_b - V_a} \right)^{\gamma-1}$, where V_a and V_b ($> V_a$) are the constant volumes along the two isochores. It is therefore appropriate to take the frequency ω as the volume variable [53] and to identify this model as a quantum version of the classical Otto refrigeration cycle.

III. THE OPTIMIZATION OF QUANTUM OTTO REFRIGERATION CYCLE

Making use of Eqs. (11), (12), and (14), we obtain the relation: $n_2 - n_1 = (n_c^{eq} - n_h^{eq}) \frac{(e^{2\gamma_c \tau_c} - 1)(e^{2\gamma_h \tau_h} - 1)}{e^{2\gamma_c \tau_c + 2\gamma_h \tau_h} - 1}$, with $n_c^{eq} = \frac{1}{2}[\tanh(\beta_c \omega_a/2) + 1]$ and $n_h^{eq} = \frac{1}{2}[\tanh(\beta_h \omega_b/2) + 1]$. When the two isochores are quasistatic ($\tau_c \rightarrow \infty$ and $\tau_h \rightarrow \infty$), the system approaches thermal equilibrium with the cold (hot) reservoir and thus n_2 (n_1) tends to be maximum (minimum) value n_c^{eq} (n_h^{eq}). The same values of n_c^{eq} (n_h^{eq}) can also be obtained in a different method [38], which is based on the assumption that the system is at thermal equilibrium. That is, for the system at thermal equilibrium with a heat reservoir at constant temperature β , the occupation probabilities p_g and p_e satisfy the Boltzmann distribution: $p_e = p_g e^{-\beta\omega}$, in which $p_e + p_g = 1$. Then occupation probabilities are given by $p_g = 1 - p_e = \frac{1}{e^{-\beta\omega} + 1}$, giving rising to $n^{eq} = \frac{1}{2}[\tanh(\beta\omega/2) + 1]$.

Casting the factor 2 into $\gamma_{h,c}$ for the expression of $(n_2 - n_1)$, we arrive at

$$n_2 - n_1 = \Delta n^{eq} f(\tau_c, \tau_h), \quad (17)$$

where we have defined $f(\tau_c, \tau_h) \equiv \frac{(e^{\gamma_c \tau_c} - 1)(e^{\gamma_h \tau_h} - 1)}{e^{\gamma_c \tau_c + \gamma_h \tau_h} - 1}$ and $\Delta n^{eq} \equiv n_c^{eq} - n_h^{eq}$. The expression for $f(\tau_c, \tau_h)$ is the same as those derived from heat engines or refrigerators within different approaches [14, 38, 47], providing a strong argument in favor of our approach. Additionally, the difference Δn^{eq} for the two-level system is the same as one obtained from a spin-1/2 system [47]. Physically, the particle in the two-level system makes transitions between the upper and lower levels by exchanging energy with the cold or hot reservoir during the interaction interval, indicating that the two-level system is in complete analogy with the spin-1/2 system. Considering Eqs. (14), (16) and (17), and the cycle time $t_{cycle} = \tau_{adi} + \tau_c + \tau_h$ with $\tau_{adi} = \tau_{hc} + \tau_{ch}$, one can write the cooling rate $R = Q_c/t_{cycle}$ and the objective function $\chi = \varepsilon R/t_{cycle}$ as

$$R = \frac{f(\tau_c, \tau_h) \omega_a \Delta n^{eq}}{\tau_{adi} + \tau_c + \tau_h}, \quad (18)$$

and

$$\chi = \frac{f(\tau_c, \tau_h) \omega_a^2 \Delta n^{eq}}{(\omega_b - \omega_a)(\tau_{adi} + \tau_c + \tau_h)}. \quad (19)$$

From Eqs. (17) and (18), the condition for the interrelation between the temperatures of heat reservoir and the frequency values can be derived as $\beta_c/\beta_h < \omega_b/\omega_a$, which must be satisfied in order that the refrigerator can do cooling and is the opposite inequality of positive work condition of the heat engine (see Eq. (24) of Ref. [38]). In the heat refrigerator work is done on the working subsystem and thus no useful work is done, thereby indicating that Carnot's bound is not violated.

The figure of merit, χ , is a product of two functions: $G(\beta_c, \omega_a, \beta_h, \omega_b) \equiv \omega_a^2 \Delta n^{eq}/(\omega_b - \omega_a)$, a function merely depends on the external parameters β and ω , and $F \equiv f(\tau_c, \tau_h)/(\tau_{adi} + \tau_c + \tau_h)$ which describes the time allocations on the isochores and adiabats. In the case when the external constraints of the refrigerator are given, optimizing the objective function χ is equivalent to optimizing the time-dependent function $F(\tau_c, \tau_h)$. Because the probabilities $\mathbf{p}(\tau_c, \tau_h)$, which determine the difference between n_2 and n_1 and thus determine the objective function χ as well as COP ε , are functions of the time allocation to the two isochores, the interaction time (τ_c or τ_h) taken for either of the two isochores as one of detailed protocols independently determine χ as well as ε . Setting $\partial F/\partial \tau_c = 0$ and

$\partial F/\partial \tau_h = 0$, the optimal time allocations on the cold and hot isochores is obtained,

$$\gamma_c[\cosh(\gamma_h \tau_h) - 1] = \gamma_h[\cosh(\gamma_c \tau_c) - 1], \quad (20)$$

which was derived much earlier in Ref. [14] and gives the optimal protocols for the refrigeration cycle. The times spent on the two isochores, τ_c and τ_h , are not independent variables as they satisfy the relation (20). This is not surprising, since the optimal protocols are fixed through optimization on the χ function as we have done. When $\gamma_c = \gamma_h$, the optimal times spent on the two isochores satisfy the relation $\tau_c = \tau_h$.

Now consider the optimization on the external constraints of the refrigerator in which the time taken for the adiabats τ_{adi} is assumed to be constant. Based on Eq. (19), optimizing the figure of merit χ becomes equivalent to optimizing two bounds of the energy unit ω_a and ω_b . Extremal conditions $\partial\chi/\partial\omega_a = 0$ and $\partial\chi/\partial\omega_b = 0$ leads to the following relations:

$$\frac{\beta_c x_c (\omega_b - \omega_a)}{x_c + 1} = \frac{x_h - x_c}{x_h + 1} \left(\frac{2\omega_b - \omega_a}{\omega_a} \right), \quad (21)$$

and

$$\frac{\beta_h x_h (\omega_b - \omega_a)}{x_h + 1} = \frac{x_h - x_c}{x_c + 1}, \quad (22)$$

where we have used $x_c \equiv e^{-\beta_c \omega_a}$ and $x_h \equiv e^{-\beta_h \omega_b}$. This set of two nonlinear equations can and only can be solved numerically to yield the optimal values of ω_a and ω_b , provided that the temperatures of two heat reservoirs β_c and β_h are given. In Fig. 2 we plot the COP at maximum χ figure of merit, ε^* , as a function of Carnot COP ε_C , comparing the CA COP $\varepsilon_{CA} = \sqrt{1 + \varepsilon_C} - 1$ with the values of $\varepsilon_+ = (\sqrt{9 + 8\varepsilon_C} - 3)/2$ (which is discussed below). Figure 2 shows that our numerical calculations (ε_N^*) are in nice agreement with the values of ε_{CA} , which were obtained previously in low-dissipation Carnot-like refrigerators under symmetric conditions [27, 30, 31] or endoreversible refrigerators with Newton's heat transfer law [24].

In the high-temperature limit when $\beta\omega \ll 1$ and thus $\tanh(\beta\omega/2) \simeq \beta\omega/2$, the heat transport law is identified as the linear phenomenological law in irreversible thermodynamics, since the amounts of heat exchanged during two isochores, given by Eq. (14) and (15), simplify to $Q_c = \gamma_c \omega_a^2 (\beta_c - \beta_h) f(\tau_c, \tau_h)/4$ and $Q_h = \gamma_h \omega_b^2 (\beta_c - \beta_h) f(\tau_c, \tau_h)/4$, respectively. In such a case, substitution of the approximation of $\tanh(\beta\omega/2) \simeq \beta\omega/2$ into Eq. (19) leads to $\chi = \frac{f(\tau_c, \tau_h) \omega_a^2 (\beta_c \omega_a - \beta_h \omega_b)}{4(\omega_b - \omega_a)(\tau_{adi} + \tau_c + \tau_h)}$. Although χ is a monotonically increasing function of ω_b , one

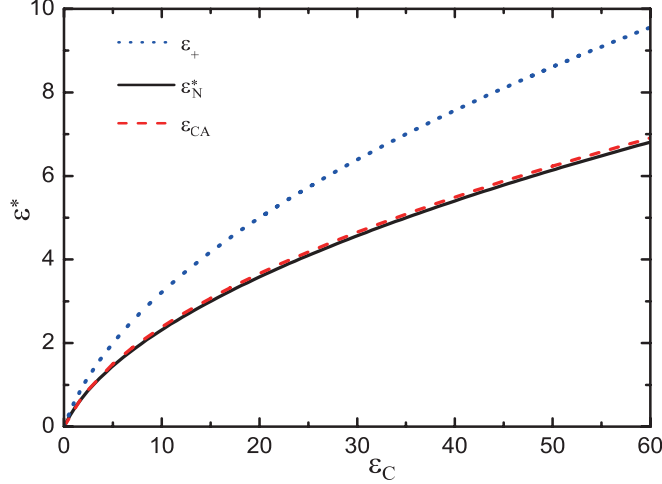


FIG. 2: (Color online) COP at maximum χ figure of merit ε^* as a function of the Carnot COP ε_C . The numerical values of the COP, ε_N^* , are denoted by a black solid line and they are in nice agreement with the values of ε_{CA} which are denoted by red dashed line. The optimal COP obtained in the high-temperature limit, ε^+ , is represented by a blue dotted line.

can optimize χ in local region at given ω_b by setting $\partial\chi/\partial\omega_a = 0$, leading to the COP at maximum χ figure of merit

$$\varepsilon^* = \varepsilon_+ \equiv (\sqrt{9 + 8\varepsilon_C} - 3)/2. \quad (23)$$

This result, reached in the high-temperature limit when the heat transport law is linear phenomenological law in irreversible thermodynamics, is particularly interesting. It is identical to a reported universal upper bound that was derived in Refs. [30, 31] using the low-dissipation assumption, and it also coincides with the upper bound obtained in a minimally irreversible refrigerator model [29]. In the high-temperature case when in which the heat transport law is linear phenomenological law, our refrigerator model reproduces the same upper bound as one derived from the low-dissipation refrigerators in the extremely asymmetric limit. This seems to imply that the same limit might be taken both for the refrigerators with the linear phenomenological law and for low-dissipation refrigerators in the asymmetric dissipation limit. These values, however, indicate greater validity to those values obtained in previous papers [29–31], as lies in the fact that they were derived from the master equation (1) based on stochastic processes.

The low-temperature limit when $\beta \gg 1$ leads to $\tanh(\beta\omega/2)$ approaches 1. It is therefore indicated that the amount of refrigeration per cycle, Q_c , becomes vanishing and the

refrigerator has lost its role in this case.

IV. COP AT MAXIMUM χ DERIVED FROM THE REFRIGERATOR MODEL UNDER THE TIGHT-COUPPLING CONDITION

In order to study further the COP at maximum χ figure of merit, we present the optimization on the performance of the cyclic refrigerator model within the framework of irreversible thermodynamics, identifying our model as a linear irreversible refrigerator that works on the linear regime, close to thermal equilibrium.

Since the working system comes back to the original state after a single cycle, the entropy production rate of the refrigeration cycle is given by $\dot{\sigma} = \beta_h \dot{Q}_h - \beta_c \dot{Q}_c$, which takes the form

$$\dot{\sigma} = \beta_c \dot{W} + \dot{Q}_h (\beta_h - \beta_c), \quad (24)$$

where the dot (\cdot) represents the physical quantity divided by the cycle period t_{cycle} . Within the context of irreversible thermodynamics, the entropy production σ can be expressed in terms of the decomposition: $\dot{\sigma} = J_1 X_1 + J_2 X_2$, where $J_1(J_2)$ denotes thermodynamic flux and $X_1(X_2)$ is the corresponding conjugate thermodynamic force. From this decomposition and Eq. (24), we define the thermodynamic fluxes [13, 19, 29]

$$J_1 \equiv 1/t_{cycle}, J_2 \equiv \dot{Q}_h, \quad (25)$$

and their conjugate thermodynamic forces

$$X_1 \equiv \beta_c W, X_2 \equiv \beta_h - \beta_c. \quad (26)$$

Theses fluxes and forces can be described by using Onsager relations as [12, 54]

$$J_1 = L_{11} X_1 + L_{12} X_2, \quad (27)$$

$$J_2 = L_{21} X_1 + L_{22} X_2, \quad (28)$$

where L_{ij} 's are the Onsager coefficients with the symmetry relation $L_{12} = L_{21}$ and they satisfy the constraints: $L_{11} \geq 0, L_{22} \geq 0, L_{11}L_{22} - L_{12}L_{21} \geq 0$. Noteworthy, linear irreversible thermodynamics described by the linear relations between fluxes and forces [see Eqs. (27) and (28)] holds well, provided that the systems are at local equilibrium and they cannot operate at rates keeping them far away from the limit of relatively near equilibrium [12, 44].

We define $q \equiv L_{12}/\sqrt{L_{11}L_{22}}$ as the usual coupling strength parameter and find $|q| \leq 1$ from these constraints. From Eqs. (27) and (28), we have

$$J_2 = \frac{L_{21}}{L_{11}}J_1 + (1 - q^2)L_{22}X_2. \quad (29)$$

The heat current absorbed from the cold reservoir, $\dot{Q}_c = \dot{Q}_h - \dot{W}$, can be expressed as $\dot{Q}_c = J_2 - X_1J_1/\beta_c$, and the COP ε for the refrigerator is given by

$$\varepsilon = \frac{\dot{Q}_c}{\dot{W}} = \frac{\beta_c J_2}{X_1 J_1} - 1, \quad (30)$$

where Eqs. (25) and (26) have been used. It follows, substituting Eq. (29) into Eq. (30), that the COP ε takes the form: $\varepsilon = \frac{\beta_c L_{12}}{X_1 L_{11}} + (1 - q^2)\frac{L_{22}X_2}{J_1} - 1$. In view of the fact that $0 < q^2 < 1$ and $X_2 = \beta_h - \beta_c (< 0)$, we find that the COP ε^* increases monotonically as the value of q^2 increases and approaches its maximum value when the tight-coupling condition is satisfied with $|q| = 1$. For the remainder of the paper, we will discuss the COP at maximum χ figure of merit for the refrigerator which fulfills the tight-coupling condition $|q| = 1$.

From Eq. (29) and the tight-coupling condition $|q| = 1$, we obtain the relation: $J_2 = \lambda J_1$ with $\lambda \equiv \frac{L_{12}}{L_{11}}$, a quantity independent of the thermodynamic forces. For the tight-coupling refrigerator, the COP and the target function, $\chi = \varepsilon \dot{Q}_c$, then become

$$\varepsilon = \frac{\beta_c \lambda}{X_1} - 1, \quad (31)$$

$$\chi = \frac{\beta_c}{X_1} \left(\lambda - \frac{X_1}{\beta_c} \right)^2 J_1 = \frac{\beta_c}{X_1} \left(\lambda - \frac{X_1}{\beta_c} \right)^2 L_{11} (X_1 + \lambda X_2), \quad (32)$$

respectively. Setting $\frac{\partial \chi}{\partial X_1} = 0$ for given inverse temperatures β_c and β_h , we obtain the physical solution at $X_1 = -(\sqrt{X_2^2 - 8\beta_c X_2} + X_2)\lambda/4$. Substituting this solution into Eq. (32), we find that the COP at maximum χ figure of merit, ε^* , is also given by Eq. (23). Note that, this optimal value of ε is also the upper bound of the COP at maximum χ figure of merit, since the refrigerator model satisfy tight-coupling condition $|q| = 1$, which gives maximum value of COP, as we discussed below Eq. (30).

Before ending this section, we should emphasize that, as in previous low-dissipation refrigerators [30, 31] and in minimally nonlinear irreversible refrigerators [29] which were also under the assumption of the local equilibrium, we re-derive the upper bound ε_+ of optimal COP from the linear irreversible refrigerators. Whether under low-dissipation assumption or within framework of irreversible thermodynamics, the phenomenological heat transfer laws

are avoided for the cyclic refrigerators. However, the upper bound of the COP at maximum χ figure of merit for refrigerators is the same as that derived in the the refrigerators with a certain heat transfer law. The issue of exploring the intrinsic and universal relation between the heat devices without use of heat transfer laws and those with certain heat transfer laws may not be easy to address, but it deserves to be studied in the future work. (Similar attempts have been made to solve such a problem, and some interesting results have been found for heat engines. See, for example, Ref. [18]).

V. CONCLUSIONS

In conclusion, we have established a quantum Otto refrigerator that consists of two isochores (two constant frequencies ω_a and ω_b) and two adiabats by using a two-level atomic system as the working substance. Employing finite-time thermodynamics, we considered the COP at maximum χ figure of merit, ε^* , for a quantum Otto refrigerator working with a two-level atomic system, optimizing χ with respect two frequencies ω_a and ω_b . Our numerical calculations show that the the values of ε^* agree very well with the CA values $\varepsilon_{CA} = \sqrt{1 + \varepsilon_C} - 1$ at finite temperatures. In the high-temperature limit, we obtained merely considering ω_a as a freedom the COP at maximum χ as $\varepsilon^* = \varepsilon_+ = (\sqrt{9 + 8\varepsilon_C} - 3)/2$, which is the upper bound of the optimal COP in low-dissipation or minimally nonlinear irreversible refrigerators. Within the framework of irreversible thermodynamics, we showed that the COP at maximum χ is also bounded from above the value of ε_+ , taking our model as a liner irreversible refrigerator.

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- [53] From another point of view, we find that the frequency ω plays the role of the volume of the system (see in Ref. [37] for more details). The frequency ω is, in fact, the energy gap $\Delta\varepsilon$ between the ground and excited states for the two state system. The frequency ω must be a function of characteristic size (volume) L of the system, and contant ω means constant volume of the sytem. That is, the isochoric process for the two level system is described by the constant-frequency process.
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